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CONSIDERATIONS ON SECOND ORDER BEAM THEORIES

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Abstract—By comparing the phase velocity predictions of various higher order beam theories with the second order approximations to the exact frequency equations for circular and thin rectangular section beams it is concluded that equivalence requires the inclusion of rotatory inertia, shear deflection, lateral contraction inertia and transverse direct stress corrections to the Euler–Bernoulli beam theory. In addition it is shown that Cowper's expression for the shear coefficient in the constitutive equation for shear is in fact identical to the original Timoshenko definition, the apparent difference arising through employing averaged rather than centre-line displacement quantities.

1. INTRODUCTION

Interest in the accuracy of second order beam theories has been recently revived by Nicholson and Simmonds[1] together with the ensuing discussion on various aspects of beam and plate theories by several authorities on elastodynamics. While the example given in [1] was restricted to static loading, albeit with an unusual body force distribution, the present work is concerned primarily with dynamic behaviour of the beam, as was Timoshenko's work[2, 3] where it was shown that if the rotatory inertia correction term to the Euler–Bernoulli theory, as introduced by Rayleigh[4], was to be included then the effect of shear deformation should also be taken into account. It is notable that the Rayleigh equation is in fact the consistent theory resulting from the Euler–Bernoulli displacement assumption; thus it has recently been argued by Leech[5] that since the elementary theory allows cross sectional rotation but does not include the inertia of such motion, then the theory is not dynamically consistent with the variational principle of motion. While such an argument is clearly justified and neglecting the rotatory inertia from the Rayleigh equation leads to the result that disturbances of infinitely small wavelength are propagated with infinitely large phase velocity, the elementary theory is valid for long wavelength low frequency propagation as will be made evident through systematic truncation of higher order terms in the exact frequency equations for circular and plane stress thin rectangular section beams.

The lateral contraction inertia term which "depends on the inertia by which cross sections are distorted in their planes" was introduced by Love[6] but has received little subsequent mention; more recently additional higher order effects, such as transverse direct stresses have been introduced[7, 8] and it is of obvious interest to know which effects must be taken into account to give the complete second order beam theory.

This would be facilitated by an asymptotic expansion for stresses and displacements for the dynamic beam but since such a theory has not been formulated at the present time, some information can be gleaned through systematic truncation to the exact frequency equations for flexural wave propagation in circular and thin rectangular section beams. Thus we first obtain the second order approximation to the Timoshenko type equation based upon consistent truncation in two small parameters ϵ and δ, defined as the ratios of a typical cross sectional dimension to wavelength and phase velocity to shear wave velocity respectively and then compare with the results of the same truncation scheme applied to the above beam cross-sections. These results are implicit in a paper by Timoshenko[3] but the procedure is shown here for the thin rectangle; the procedure for the circle is similar in form but considerably more complicated and is outlined by Pochhammer[9]. This comparison shows that equivalence of phase velocities for long wavelength low frequency propagation is obtained when "shear" coefficients $K = 5(1+\nu)/(6+5\nu)$ for the rectangle and $K = 6(1+\nu)^2/(7+12\nu+4\nu^2)$ for the circle are employed in the coefficient $(1+(E/KG)$ of the $\partial^4 u/\partial^2 z \partial t^2$ term of the Timoshenko type equation.

By taking these phase velocities as THE second order phase velocity approximations it is shown on the basis of previous work for circles and thin rectangles which terms are necessary for the complete second order approximation, these being rotatory inertia, shear deformation, transverse direct stresses and lateral contraction inertia.

Consideration is also given to the use of integrated rather than centre-line displacement quantities, these being employed in beam theories by Prescott[10], Cowper[11] and Stephen and Levinson[8]. It is shown that the Cowper constitutive equation for shear, which employs integrated quantities, is in fact directly equivalent to the original Timoshenko expression for shear which employed centre line quantities, and it is argued that it is preferable to employ the integrated description for displacements as one does for stresses, i.e. bending moment and shearing force.

2. FIRST AND SECOND ORDER APPROXIMATIONS

The usual method of generating higher order terms in beam, plate and shell theories is the expansion of stresses or displacements in terms of a "thickness" parameter based on the ratio of a cross sectional dimension to the length, or in the case of shells, the radius; for dynamic beam theories a glance at a typical phase velocity dispersion diagram[12], for the lowest truely flexural mode shows a second parameter, the ratio of phase velocity to shear wave velocity, to be of the same order of magnitude in the region of interest. Thus second order approximations valid for low frequency long wavelength disturbance propagation are obtained by consistent truncation in the two parameters ϵ and δ.

2.1 *Approximation to Timoshenko type equation*

The standard Timoshenko type equation[2] may be written as

$$EI_y u^{iv} + \rho A\ddot{u} - \rho I_y\left(1+\frac{E}{KG}\right)\ddot{u}'' + \frac{\rho^2 I_y \ddot{\ddot{u}}}{KG} = 0 \tag{1}$$

where prime and dot denote differentiation with respect to the axial coordinate and time respectively, and E is Young's modulus; G, shear modulus; I_y, second moment of area; ρ, density; A, cross sectional area; K, shear coefficient and u, transverse displacement of beam centre line.

If displacement u is assumed harmonic in time and axial co-ordinate, i.e.

$$u \sim e^{ia(z-c_p t)} \tag{2}$$

where c_p is phase velocity; a, wave number $= \pi/\lambda$ and λ, half wavelength and introducing the two small parameters

$$\epsilon = \frac{r_g}{\lambda}, \qquad \delta = \frac{c_p}{c_s}$$

eqn (1) becomes

$$K\left(\frac{c_\epsilon}{c_s}\right)^2 \pi^2\epsilon^2 - K\delta^2 - \pi^2\epsilon^2\left\{K + \left(\frac{c_\epsilon}{c_s}\right)^2\right\}\delta^2 + \pi^2\epsilon^2\delta^4 = 0 \tag{3}$$

where $c_\epsilon = \sqrt{(E/\rho)}$ = rod velocity; $c_s = \sqrt{(G/\rho)}$ = shear velocity; $r_g = \sqrt{(I_Y/A)}$ = radius of gyration.

Since the small parameters ϵ and δ are of the same order of magnitude which is denoted Δ, the first approximation is obtained by truncating terms greater than Δ^2, when eqn (3) becomes

$$\left(\frac{c_\epsilon}{c_s}\right)^2 \pi^2\epsilon^2 = \delta^2 \tag{4}$$

which will be seen to be equivalent to

$$c_p = c_e \pi \frac{r_g}{\lambda} \; ; \tag{5}$$

this is the phase velocity for the Euler-Bernoulli beam.

The second order approximation is obtained by truncating terms greater than Δ^4 when eqn (3) gives

$$c_p^2\left[1+\left(\frac{\pi r_g}{\lambda}\right)^2\left(1+\frac{E}{KG}\right)\right]=c_e^2\left(\frac{\pi r_g}{\lambda}\right)^2. \tag{6}$$

Taking the square root and expanding by the binomial theorem gives

$$\frac{c_p}{c_e}=\frac{\pi r_g}{\lambda}\left[1-\frac{\pi^2 r_g^2}{\lambda^2}\left(1+\frac{E}{KG}\right)+\frac{3}{8}\frac{\pi^4 r_g^4}{\lambda^4}\left(1+\frac{E}{KG}\right)^2-\cdots\right] \tag{7}$$

and the second order phase velocity approximation becomes

$$c_p = c_e\frac{\pi r_g}{\lambda}\left[1-\frac{\pi^2 r_g^2}{2\lambda^2}\left(1+\frac{E}{KG}\right)\right] \tag{8}$$

2.2 *Approximation to plane stress solution for thin rectangle*

Cowper[13] gives the frequency equation for the thin rectangular beam Fig. 1 as

$$2(1-\nu)\alpha\beta\tanh\alpha=\left(\left(\frac{\pi h}{L}\right)^2+\alpha^2\right)\left(\left(\frac{\beta L}{\pi h}\right)^2-\nu\right)\tanh\beta \tag{9}$$

where

$$\alpha^2=\left(\frac{\pi h}{L}\right)^2-\frac{2(1+\nu)\rho\omega^2 h^2}{E};$$

$$\beta^2=\left(\frac{\pi h}{L}\right)^2-\frac{(1-\nu)^2\rho\omega^2 h^2}{E};$$

and ω is natural frequency (rad/sec) and L = length of beam = half wavelength for fundamental mode.

If we define as small quantities

$$\epsilon=\left(\frac{\pi h}{L}\right),\qquad \delta=\frac{c_p}{c_s}$$

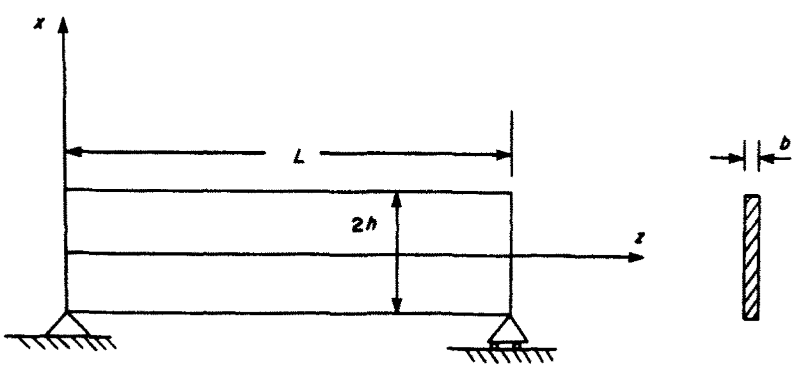

Fig. 1. Thin rectangular beam.

then

$$\alpha^2 = \epsilon^2(1-\delta^2), \qquad \beta^2 = \epsilon^2\left(1-\left(\frac{1-\nu}{2}\right)\delta^2\right) \tag{10}$$

and the frequency equation becomes

$$4(1-\delta^2)^{1/2}\left(1\left(\frac{1-\nu}{2}\right)\delta^2\right)\tanh\{\epsilon(1-\delta^2)^{1/2}\} = (2-\delta^2)^{1/2}\tanh\left\{\epsilon\left(1-\left(\frac{1-\nu}{2}\right)\delta^2\right)^{1/2}\right\}. \tag{11}$$

Employing three terms of the ascending series for the hyperbolic tangent and cancelling the term $\epsilon(1-((1-\nu)/2)\,\delta^2)^{1/2}$ gives

$$4(1-\delta^2)\left[1-\frac{\epsilon^2}{3}(1-\delta^2)+\frac{2\epsilon^4}{15}(1-\delta^2)^2\right]$$
$$= (2-\delta^2)^2\left[1-\frac{\epsilon^2}{3}\left(1-\left(\frac{1-\nu}{2}\right)\delta^2\right)+\frac{2\epsilon^4}{15}\left(1-\left(\frac{1-\nu}{2}\right)\delta^2\right)^2\right]. \tag{12}$$

By expanding (12) and truncating terms greater than Δ^2 the first order approximation is obtained as

$$2\epsilon^2(1+\nu) = 3\delta^2 \qquad \text{or} \qquad \omega^2 = \frac{1}{3}\frac{\pi^4 h^2}{L^4}\frac{E}{\rho}. \tag{13}$$

This is the fundamental natural frequency for the Euler–Bernoulli model of the beam.

For the second order approximation we truncate terms greater than Δ^4 to give

$$c_p^2 = \frac{\epsilon^2}{3}c_\epsilon^2\left(1-\frac{4}{5}\epsilon^2\right)\left(1+\epsilon^2\left(\frac{1+2\nu}{3}\right)\right)^{-1}. \tag{14}$$

Taking the square root nad expanding by the Binomial theorem gives

$$c_p = \frac{\epsilon}{\sqrt{3}}c_e\left(1-\frac{2}{5}\epsilon^2-\frac{2\epsilon^4}{25}\cdots\right)\left(1-\epsilon^2\frac{(1+2\nu)}{6}+\frac{(1+2\nu)}{24}\epsilon^4\cdots\right). \tag{15}$$

again neglecting terms smaller than Δ^2 of the largest we find

$$c_p = \frac{c_\epsilon}{\sqrt{3}}\frac{\pi h}{L}\left[1-\frac{\pi^2 h^2}{6L^2}\left(\frac{12}{5}+1+2\nu\right)\right]. \tag{16}$$

Comparison with eqn (8), with $r_g = (h/\sqrt{3})$ shows that equivalence requires

$$1+\frac{E}{KG} = \frac{12}{5}+1+2\nu$$

or

$$K = \frac{5(1+\nu)}{6+5\nu}. \tag{17}$$

It is thus concluded that a Timoshenko type equation with $K(5(1+\nu)/6+5\nu)$ will include all second order effects apparent for the thin (plane stress) rectangle.

2.3 *Approximations to exact frequency equation for circular section beam*

Pochhammer[9] obtains the exact frequency equation for a circular section rod of infinite length and gives the second order phase velocity approximation as

$$c_p = c_e \frac{\pi r_g}{\lambda}\left[1 - \frac{\pi^2 r_g^2}{3\lambda^2}\left(\frac{7}{2} + \frac{E}{G} - \frac{G}{E}\right)\right] \tag{18}$$

where the same truncation scheme as above is employed.

Agreement with the Timoshenko type second order approximation (8) requires

$$\frac{1}{3}\left(\frac{7}{2} + \frac{E}{G} - \frac{G}{E}\right) = \frac{1}{2}\left(1 + \frac{E}{KG}\right)$$

or

$$K = 6(1+\nu)^2/(7 + 12\nu + 4\nu^2). \tag{19}$$

Thus it is concluded that a Timoshenko type equation with the above value for K will include all second order effects apparent for the circular section.

3. INTEGRATED DISPLACEMENT QUANTITIES IN BEAM THEORIES

The original Timoshenko beam theory employed two dependent variables u and ψ which are centre-line transverse displacement and centre-line cross sectional rotation respectively; in more recent beam theories[8, 10, 11] the two integrated displacement quantities $U = (1/A)\int u \, dA$, $\Psi = (1/I_y)\int xw \, dA$ are employed.

We first derive the relationship between the two sets of variables for St-Venant flexure displacement distribution (the forms of the relationships are similar for other loading configurations) and then consider the effect of employing integrated rather than centre line quantities on the original constitutive relationship for shear given by Timoshenko.

For the tip loaded cantilever (Fig. 2), Love[14] gives the transverse displacement as

$$u = \frac{Q}{EI_y}\left[\left(\frac{L-z}{2}\right)\nu(x^2 - y^2) + \frac{Lz^2}{2} - \frac{z^3}{6}\right] \tag{20}$$

and writing $u_0 = u|_{x=y=0}$ for the centre-line displacement gives

$$u_0 = \frac{Q}{EI_y}\left[\frac{Lz^2}{2} - \frac{z^3}{6}\right]. \tag{21}$$

The integrated displacement is

$$U = \frac{1}{A}\iint u \, dx \, dy = \frac{Q}{EI_y}\left[\left(\frac{L-z}{2}\right)\nu\frac{(I_y - I_x)}{A} + \frac{Lz^2}{2} - \frac{z^3}{6}\right] \tag{22}$$

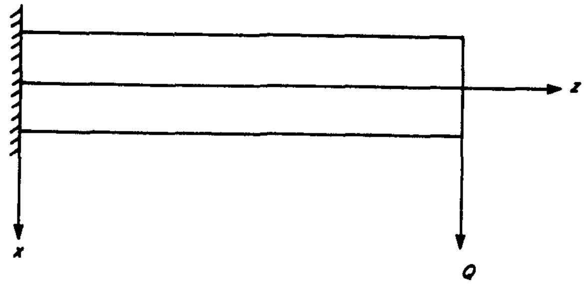

Fig. 2. Tip loaded cantilever beam.

and hence

$$\frac{\partial U}{\partial z}=\frac{\partial u_0}{\partial z}-\frac{Q\nu}{2EI_yA}(I_y-I_x). \tag{23}$$

Again from Love[14], the longitudinal displacement w is

$$w=-\frac{Q}{EI_y}\left[x\left(Lz-\frac{z^2}{2}\right)+\chi+xy^2\right] \tag{24}$$

where χ is the flexure function. The centre-line sectional rotation

$$\psi=\left.\frac{\partial w}{\partial x}\right|_{x=y=0}$$

and from (24) we have

$$\Psi=-\frac{Q}{EI_y}\left[\left(Lz-\frac{z^2}{2}\right)+\left.\frac{\partial\chi}{\partial x}\right|_{x=y=0}\right]. \tag{25}$$

The integrated rotational displacement is $\Psi=(1/I_y)\iint xw\,\mathrm{d}x\,\mathrm{d}y$ giving

$$\Psi=-\frac{Q}{EI_y}\left(Lz-\frac{z^2}{2}\right)-\frac{Q}{EI_y^2}\iint x(\chi+xy^2)\,\mathrm{d}x\,\mathrm{d}y \tag{26}$$

and hence

$$\Psi=\psi+\frac{Q}{EI_y}\left[\left.\frac{\partial\chi}{\partial x}\right|_{x=y=0}-\frac{1}{I_y}\iint x(\chi+xy^2)\,\mathrm{d}x\,\mathrm{d}y\right]. \tag{27}$$

According to the original Timoshenko paper[2]

$$Q=K_TAG\left(\frac{\partial u_0}{\partial z}+\psi\right) \tag{28}$$

where K_T is defined as the ratio of average shear stress to shear stress at the centroid. For St-Venant flexure stress distribution the coefficient is

$$K_T=\frac{-2(1+\nu)I_y}{A\left.\frac{\partial\chi}{\partial x}\right|_{x=y=0}} \tag{29}$$

although Timoshenko employed approximate shear stress distributions in the above definition giving $K=(2/3)$ for the rectangle and $K_T=(3/4)$ for the circle.

If in (28) ψ is replaced by Ψ we have

$$Q=-\frac{2(1+\nu)I_y}{\frac{A}{I_y}\iint x(\chi+xy^2)\,\mathrm{d}x\,\mathrm{d}y}\cdot AG\left(\frac{\partial u_0}{\partial z}+\Psi\right) \tag{30}$$

and further replacing the centre-line transverse displacement u_0 by the integrated quantity U gives

$$Q=K_cAG\left(\frac{\partial U}{\partial z}+\Psi\right) \tag{31}$$

where K_c is identical to the shear coefficient derived by Cowper[11] and is given by

$$K_c = \frac{2(1+\nu)I_y}{\frac{\nu}{2}(I_x - I_y) - \frac{A}{I_y}\iint x(\chi + xy^2)\,\mathrm{d}x\,\mathrm{d}y}. \tag{32}$$

Thus the difference between the Cowper coefficient and the original Timoshenko coefficient is directly attributable to the employment of integrated rather than centre-line displacements.

Having demonstrated one of the mathematical effects of employing integrated rather than centre-line quantities it remains to discuss which of the two descriptions is preferable; this question is allied to the problem of whether overall or point-wise behaviour of the beam should be considered as the validity criteria, in that improved correlation between approximate theory and exact phase velocity predictions may not necessarily result in improved correlation for stresses and displacements.

Since one of the essential features of every technical beam theory is the use of integrated stress quantities *viz* bending moment and shearing force it is arguable that consistency requires the use of integrated displacement quantities; this is particularly valid at the beam ends where point-wise displacement behaviour is rarely known. The integrated displacement quantities arise naturally in the derivation of the beams equations[8, 11] through integration of the elasticity equilibrium equations and since the resulting theory[8] has been shown to give agreement, to second order, with the exact theory in terms of overall beam behaviour, i.e. frequency and phase velocity, the use of integrated quantities appears preferable. Conversely, if some point-wise quantity, e.g. maximum strain, is required this can be obtained from the integrated displacement using expressions(20) and (23) or alternatively, since it has been shown[15] that the assumption of stresses and displacements in the vibrating beam to be identical to those in the beam subjected to uniform body force loading provides agreement, to second order, with the exact theory it appears reasonable to obtain such point-wise quantities from the exact static theory rather than the approximate technical vibration theory. From this albeit brief discussion the author is led to conclude that the integrated displacement description is preferable.

4. REMARKS ON PREVIOUS SECOND ORDER EQUATIONS

Of previous second order beam theories the difference in form can be seen from the equations given by:

Rayleigh[4]:

$$EI_y u^{iv} + \rho A\ddot{u} - \rho I_y \ddot{u}'' = 0. \tag{33}$$

Love[6]

$$EI_y u^{iv} + \rho A\ddot{u} - \rho I_y \ddot{u}''\left(1 + \nu\frac{(I_x - I_y)}{I_y}\right) = 0. \tag{34}$$

Timoshenko[2]:
(Type)

$$EI_y u^{iv} + \rho A\ddot{u} - \rho I_y \ddot{u}''\left(1 + \frac{E}{KG}\right) + \frac{\rho^2 I_y}{KG}\ddot{\ddot{u}} = 0. \tag{35}$$

Stephen and Levinson[8]:

$$EI_y u^{iv} + \rho A\ddot{u} - \rho I_y \ddot{u}''\left(1 + \frac{E}{K_1 G}\right) + \frac{\rho^2 I_y \ddot{\ddot{u}}}{K_2 G} = 0. \tag{36}$$

These are compared with the second order approximation to the Timoshenko equation (35) which is equivalent to

$$EI_y u^{iv} + \rho A\ddot{u} - \rho I_y \ddot{u}''\left(1 + \frac{E}{KG}\right) = 0. \tag{37}$$

The relationship between the Rayleigh equation (33) and (37) is immediately evident. The Love equation, which is a Rayleigh equation modified to include lateral contraction inertia will be equivalent to (37) if we put

$$K = \frac{2(1+\nu)I_y}{\nu(I_x - I_y)}. \tag{38}$$

Of the various Timoshenko type equations of form (35) we mention the Cowper equation where the coefficient has been given in eqn (32) and comparison with (38) shows the inclusion of one half of the lateral contraction term. If we now take the coefficient according to eqn (30) as accounting for non uniform shear stress distribution then clearly the Cowper coefficient accounts for shear and partly for lateral contraction inertia. A coefficient obtained by the present author[15], by assuming the stress distribution in a beam performing long wavelength flexural vibration to be equivalent to the distribution in a beam subjected to constant body force loading, thus including direct transverse stresses, and identical to K_1 in (36) may be written as

$$K = \frac{2(1+\nu)I_y}{\left\{\nu(I_x - I_y) - \frac{A}{I_y}\iint x(\chi + xy^2)\,dx\,dy - \frac{\nu A}{2(1+\nu)I_y}\iint xy\left(\frac{\partial\chi}{\partial y} + (2+\nu)xy\right)dx\,dy - \frac{\nu A}{2(1+\nu)I_y}\iint\left(\frac{x^2-y^2}{2}\right)\left(\frac{\partial\chi}{\partial x} + \frac{\nu x^2}{2} + \left(\frac{2-\nu}{2}\right)y^2\right)dx\,dy\right\}} \tag{39}$$

and the terms in the denominator can be identified according to the second order effect. Thus the first two terms account for lateral contraction inertia and shear respectivity while the second two terms account for direct transverse stress distribution.

This dependence is more evident by sub-dividing the coefficient according to the second order effect being taken into account. Thus we have

$$\frac{1}{K} = \frac{1}{K_a} + \frac{1}{K_b} + \frac{1}{K_c} \tag{40}$$

where

$$K_a = \frac{2(1+\nu)I_y}{\nu(I_x - I_y)},$$

accounts for lateral contraction inertia;

$$K_b = \frac{2(1+\nu)I_y}{-\frac{A}{I_y}\iint x(\chi + xy^2)\,dx\,dy}$$

accounts for shear distribution and

$$K_c = \frac{2(1+\nu)I_y}{\frac{-\nu A}{2(1+\nu)I_y}\iint\left\{\left(xy\left(\frac{\partial\chi}{\partial y} + (2+\nu)xy\right) + \left(\frac{x^2-y^2}{2}\right)\left(\frac{\partial\chi}{\partial x} + \frac{\nu x^2}{2} + \left(\frac{2-\nu}{2}\right)y^2\right)\right)\right\}dy\,dy},$$

accounts for direct transverse stress distribution.

Expression (39) for the coefficient has been evaluated in[15] for several beam cross sections and yields the values

$$K = 6(1+\nu)^2/(7 + 12\nu + 4\nu^2) \quad \text{and} \quad K = 5(1+\nu)/(6 + 5\nu)$$

for the circle and thin rectangle respectively; since it has been shown that these values account for all second order effects according to the truncation scheme employed it is reasonable to assume that (39) will account for all second order effects in arbitrary symmetric sections. In addition expression (39) provides

$$K = (1+\nu)/(2+\nu)$$

for the thin walled circular tube and gives, to second order, the correction to elementary beam theory frequencies as does shell theory[16]. Confidence in the suitability of the above values is reinforced by the work of Kaneko[17] who reviewed various theoretical and empirical expressions for the coefficient and having made a comparison with published experimental results concluded that these values afforded best agreement of phase velocities.

5. CONCLUSIONS

On the basis of a consistent truncation procedure applied to the exact frequency equations for a thin (plane stress) rectangular beam and a circular section beam and comparison with the predictions of various second order beam theories, it is shown that agreement of phase velocity predictions at long wavelength low phase velocity requires inclusion of the effects of shear deflection (more precisely shear curvature), lateral contraction inertia and transverse direct stress together with the rotatory inertia correction to the Euler–Bernoulli beam theory.

In addition it is concluded that integrated, rather than center-line, displacement quantities are preferable in approximate mathematical description of the beam and that required pointwise behaviour of the vibrating beam, to second order accuracy, may be found by considering the uniform body force, i.e. gravity, loading of the beam.

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